

BOUNDEDNESS IN PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we investigate bounds for solutions of perturbed nonlinear differential systems.

1. Introduction

The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : Lyapunov's second method, the method of variation of constants formula, and the use of inequalities.

The notion of h -stability (hS) was introduced by Pinto [11, 12] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h -systems.

The main conclusion to be drawn from this paper is that the use of inequalities provides a powerful tool for obtaining bounds for solutions. To do this we need some integral inequalities.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$(2.1) \quad x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and \mathbb{R}^n is the Euclidean n -space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and

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is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$(2.2) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0$$

and

$$(2.3) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2).

We recall some notions of h -stability [12].

DEFINITION 2.1. The system (2.1)(the zero solution $x = 0$ of (2.1)) is called an h -system if there exist a constant $c \geq 1$, and a positive continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c|x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough(here $h(t)^{-1} = \frac{1}{h(t)}$).

DEFINITION 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called h -stable(hS) if there exists $\delta > 0$ such that (2.1) is an h -system for $|x_0| \leq \delta$ and h is bounded.

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [6].

DEFINITION 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$(2.4) \quad \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [6].

We give some related properties that we need in the sequel.

LEMMA 2.4. [13] *The linear system*

$$(2.5) \quad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (respectively h -stable) if and only if there exist $c \geq 1$ and a positive continuous (respectively bounded) function h defined on \mathbb{R}^+ such that

$$(2.6) \quad |\phi(t, t_0)| \leq c h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.5).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$(2.7) \quad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.5. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

THEOREM 2.6. [3] *If the zero solution of (2.1) is hS , then the zero solution of (2.2) is hS .*

THEOREM 2.7. [4] *Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.2) is hS , then the solution $z = 0$ of (2.3) is hS .*

LEMMA 2.8. (Bihari-type inequality) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s)w(u(s))ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda(s)ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 2.9. [2] Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that for some $c > 0$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) w(u(s)) ds + \int_{t_0}^t \lambda_2(s) \left(\int_{t_0}^s \lambda_3(\tau) w(u(\tau)) d\tau \right) ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)) ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $u > 0$, $u_0 > 0$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)) ds \in \text{dom} W^{-1} \right\}.$$

3. Main results

In this section, we investigate bounds for the nonlinear differential systems.

THEOREM 3.1. Let $a, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v} w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that the solution $x = 0$ of (2.1) is hS with a nondecreasing function h and the perturbed term g in (2.7) satisfies

$$|\Phi(t, s, z)g(t, z)| \leq a(s)(w(|z|) + \int_{t_0}^s k(\tau)w(|z|)d\tau), \quad t \geq t_0 \geq 0,$$

where $\int_{t_0}^\infty k(s)ds < \infty$. Then any solution $y(t) = y(t, t_0, y_0)$ of (2.7) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau) d\tau \right) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau) d\tau \right) ds \in \text{dom} W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.7), respectively. Applying Lemma 2.5 and the increasing property of the function h , we obtain

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))g(s, y(s))| ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t a(s) \left[w(|y(s)|) + \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau \right] ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t a(s)h(t)w\left(\frac{|y(s)|}{h(s)}\right)ds \\ &\quad + \int_{t_0}^t a(s) \int_{t_0}^s h(t)k(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.9, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau)d\tau \right) \right], \quad t_0 \leq t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and the theorem is proved. \square

REMARK 3.1. Letting $k(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.1 in [9].

Also, we examine the bounded property for the perturbed system

$$(3.1) \quad y' = f(t, y) + \int_{t_0}^t g(s, y(s))ds, \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$, of (2.1).

THEOREM 3.2. Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.1) is hS with the increasing function h , and g in (3.1) satisfies

$$\left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| \leq a(s)w(|y(s)|) + b(s) \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau,$$

where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$ and $\int_{t_0}^\infty k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (3.1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) \int_{t_0}^s k(\tau)d\tau) ds \right],$$

where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + b(s)) \int_{t_0}^s k(\tau) d\tau ds \in \text{dom} W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solutions of (2.1) and (3.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is hS. Using two Lemma 2.4 and 2.5, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left| \int_{t_0}^s g(\tau, y(\tau)) d\tau \right| ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) a(s) w\left(\frac{|y(s)|}{h(s)}\right) ds \\ &\quad + \int_{t_0}^t c_2 h(t) b(s) \int_{t_0}^s k(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau ds. \end{aligned}$$

since h is increasing. Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.9 yields

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s)) \int_{t_0}^s k(\tau) d\tau ds \right],$$

where $c = c_1 |y_0| h(t_0)^{-1}$. The above estimation implies the boundedness of $y(t)$, and the proof is complete. \square

REMARK 3.2. Letting $k(t) = 0$ in Theorem 3.2, we obtain the same result as that of Theorem 3.2 in [9].

We need the lemma to prove the following theorem.

LEMMA 3.3. Let $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u and $u \leq w(u)$. Suppose that for some $c \geq 0$,

$$(3.2) \quad u(t) \leq c + \int_{t_0}^t \left(p(s) \int_{t_0}^s \left(q(\tau)u(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)w(u(a)) da \right) d\tau \right) ds, \quad t \geq t_0.$$

Then

$$(3.3) \quad u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \left(p(s) \int_{t_0}^s \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \left(p(s) \int_{t_0}^s \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \in \text{dom}W^{-1} \right\}.$$

Proof. Setting

$$z(t) = c + \int_{t_0}^t \left(p(s) \int_{t_0}^s \left(q(\tau) u(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) w(u(a)) da \right) d\tau \right) ds,$$

we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= p(t) \int_{t_0}^t \left(q(\tau) u(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) w(u(a)) da \right) d\tau \\ (3.4) \quad &\leq p(t) \int_{t_0}^t \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) w(u(\tau)) d\tau \\ &\leq \left[p(t) \int_{t_0}^t \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right] w(z(t)), \quad t \geq t_0, \end{aligned}$$

since $z(t)$ and $w(u)$ are nondecreasing and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$(3.5) \quad z(t) \leq c + \int_{t_0}^t \left(p(s) \int_{t_0}^s \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau w(z(s)) \right) ds.$$

It follows from Lemma 2.8 that (3.5) yields the estimate (3.3). □

THEOREM 3.4. *Let $w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u , $u \leq w(u)$, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (2.1) is an h -system with a positive continuous function h and g in (3.1) satisfies*

$$(3.6) \quad |g(t, y)| \leq a(t)(|y(t)| + \int_{t_0}^t k(s)w(|y(s)|)ds), \quad t \geq t_0, \quad y \in \mathbb{R}^n$$

where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous with

$$\int_{t_0}^\infty \frac{1}{h(s)} \int_{t_0}^s \left(a(\tau) \left(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \right) d\tau \right) ds < \infty,$$

for all $t_0 \geq 0$, then any solution $y(t) = y(t, t_0, y_0)$ of (3.1) satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t \frac{c_2}{h(s)} \int_{t_0}^s a(\tau) \left(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \right) d\tau ds \right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \frac{c_2}{h(s)} \int_{t_0}^s a(\tau) \left(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \right) d\tau ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solutions of (2.1) and (3.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is an h -system, the solution $v = 0$ of (2.2) is an h -system. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is an h -system. Applying Lemma 2.5 and (3.6), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \int_{t_0}^s |g(\tau, y(\tau))| d\tau ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 \frac{h(t)}{h(s)} \left(\int_{t_0}^s h(\tau) a(\tau) \frac{|y(\tau)|}{h(\tau)} \right. \\ &\quad \left. + \int_{t_0}^s a(\tau) \int_{t_0}^{\tau} h(r) k(r) w\left(\frac{|y(r)|}{h(r)}\right) dr \right) d\tau ds. \end{aligned}$$

Using Lemma 3.3 with $u(t) = |y(t)|h(t)^{-1}$, we obtain

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t \frac{c_2}{h(s)} \int_{t_0}^s a(\tau) \left(h(\tau) + \int_{t_0}^{\tau} h(r)k(r)dr \right) d\tau ds \right],$$

$t_0 \leq t < b_1$, where $c = c_1 |y_0| h(t_0)^{-1}$. Hence, the proof is complete. \square

REMARK 3.3. Letting $k(s) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 2.5 in [10]

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